Relationships Between the Diagonals of Regular Polygons

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Abstract

The purpose of this project is to discover how the amount of sides of a regular polygon affects the ratio between the polygon and a similar one formed from its diagonals. This has relevance to architects as they build structures more efficiently, improving folding techniques for polygons in origami, and create new compact products to conduct medicinal procedures. If the number of sides of a polygon is related to the ratio between a regular polygon and a similar polygon formed from non-opposite, diagonal lines of the original regular polygon, then the ratio will change as the number of sides changes based on a formula. The ratio between regular polygons based on non-opposite, diagonal lines of a regular polygon to form similar, inner polygons were determined by creating the formula to determine the ratio and comparing that with the measured values. The approximate side length ratios for compression level 1 were 1 for the triangle, 1 for the square, 0.382 for the pentagon, 0.577 for the hexagon, 0.247 for the heptagon, and 0.414 for the octagon. The approximate side length ratios in compression level 2 were 1 for the pentagon, 1 for the hexagon, 0.692 for the heptagon, and 0.765 for the octagon. The approximate side length ratios for compression level 3 were 1 for both the heptagon and octagon. The results showed that the hypothesis was supported because every ratio for a given compression level follows a certain formula by a method that can be used by all regular polygons. Possible errors may include making generalizations of certain properties of the diagonals and overlooking possible generalizations and properties of polygons beyond the octagon. Future research could focus on the side length ratios for any polygon or 3-dimensional figures.

Introduction

Regular polygons are shapes that have fascinated mathematicians dating back to the Greek age. Euclid's geometric discoveries in his book Elements brought upon simple, approximate constructions of regular polygons using a compass and an unmarked straightedge. The Renaissance age brought upon artists and mathematicians that improved the accuracy of drawing these intricate shapes, with Leonardo da Vinci's work on the Vitruvian Man and its heptagonal relationships as well as Durer's books on the analysis of lines, planes, and solids. In the analysis of regular polygons, formulas and functions are used to relate the angles between the sides and diagonals of the polygon with its corresponding length. Trigonometry, the Law of Sines and Cosines, radian measure in circles, and circle properties are all used in the analysis of regular polygons, given its special nature of having all side lengths and angles congruent to each other. Regular polygons are most prominent in nature, from the rocks, flowers, and animals that sustain our environment to the artistic, aesthetic elements that make regular polygons special in certain pieces of art. In this experiment, regular polygons are investigated to find smaller, similar, hidden regular polygons by drawing the diagonals of the polygon. By finding the ratio between the side lengths of all possible small polygons within the bigger polygon, an optimal ratio may be found between the common polygons, information that may be useful to improve polygonal folding techniques in origami or develop newer compact products to improve medicinal procedures.

Research Question

How does the amount of sides of a regular polygon affect the length of a similar polygon formed from non-opposite, diagonal lines of the original, regular polygon?

Methods

How to Construct the Formula for the Ratio Between a Regular Polygon's Side Length and a Regular Polygon Formed From its Diagonals:

1. Draw a regular polygon (length and number of sides doesn't matter but the polygon must have at least 7 sides (a heptagon)).
2. Draw lines between 2 points for every possible set of 2 vertices of the polygon. For even-number-sided polygons, do not draw to opposite points of the polygon.
3. Be able to prove a similar regular polygon will always exist by drawing the diagonals (See Appendix for the full proof.).
4. Generalize the angle formed from the difference of an overall interior angle of a vertex and the remaining angle formed from a diagonal that forms the smaller regular polygon (i.e. θ) based on properties of circles and the amount of sides 2 diagonals of same length that intersect at one vertex intercept.
5. Draw 3 diagonals, one that crosses at least 3 vertices, and 2 other diagonals that are drawn such that they cross the same amount of vertices as the first diagonal and intercept one vertex that is exactly one side away from the original diagonal, both cross a vertex that the original diagonal crossed.
6. Using the Law of Sines, use θ and α, the angle of each interior angle of the polygon, to find the lengths of the 2 triangles created by the diagonals in terms of one side of the regular polygon.
7. Draw 2 diagonals, one that uses one of the diagonals from the previous 3 diagonals, and another such that one end of the line is space one side away from the first diagonal and

the other end intersects the first diagonal at the opposite end of the first diagonal.

8. Using the Law of Sines, use θ and properties of a circle to find another angle to solve for the side lengths of the triangle formed from the diagonals created in step 7 in terms of a side of the regular polygon.
9. Notice that in step 6, there was an overlap between the 2 triangles of one of their sides, which forms a side of the smaller polygon, proving that this is a side of the smaller polygon using properties of circles and angle α.
10. Since the 2 triangles in step 6 are congruent, the 2 side lengths minus the overlap is the overall diagonal that was used in steps 5-8. Using the information found in step 8, set an equation that connects the side lengths of one of the sides of the two triangles in step 6 used to help form one of the overall diagonals and use the overall length of the diagonal in step 8.
11. Substitute these values in the equation in terms of y as they could be found in step 6 and 8.
12. Using algebra, move the variables around to get the ratio of the side lengths of the original regular polygon and the smaller polygon.

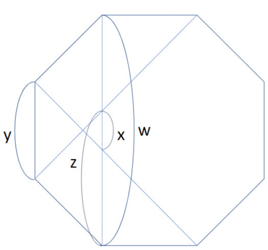

Fig. 1. Diagram for Step 4 of Procedures

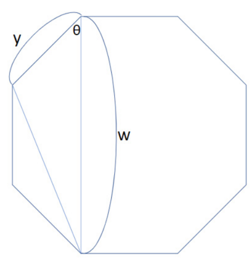

Fig. 2. Diagram for Step 6 of Procedures

Results

The Formula for the Ratio Between a Regular Polygon's Side Length and a Regular Polygon Formed From its Diagonals

$$\frac{x}{y} = \frac{2sin(\frac{360}{n} + \theta)}{sin(\frac{360}{n})} - \frac{sin(\frac{180}{n} + \theta)}{sin(\frac{180}{n})}$$

Fig. 3. The ratio x (the length of the regular polygon formed from drawing its diagonals) to y (the length of the original regular polygon), where n is the amount of sides in the regular polygon, and represents the angle formed from the difference of an overall interior angle of a vertex and the remaining angle formed from the diagonal and the remaining portion of the polygon)

Data Tables

See figures 4-5

Graph

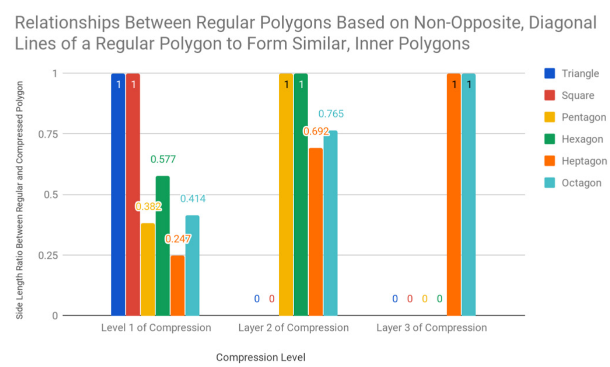

Fig. 6. This graph depicts how the ratio of side lengths between the 2 polygons vary with a given side length labeled to the right of the graph under different compression levels.

Exact/Approximate Values of the Side Length Ratio Between Regular and Compressed Polygons						
Compression Level	The Type of Regular Polygon					
	Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon
Level 1 of Compression	1	1	2-Φ* (approx. 0.382)	$\frac{\sqrt{3}}{3}$ (approx. 0.577)	$4\cos^2(\frac{\pi}{7})$ -3 (approx. 0.247)	$\sqrt{2} - 1$ (approx. 0.414)
Layer 2 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1	$2\cos(\frac{\pi}{7}) - \sec(\frac{\pi}{7})$ (approx. 0.692)	$\sqrt{2 - \sqrt{2}}$ (approx. 0.765)
Layer 3 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1

(DNE= Does Not Exist)
(*Phi or Φ is defined to be the golden ratio or $\frac{1+\sqrt{5}}{2}$)

Fig. 4. The data table depicts the calculated values of the side length ratio between the two regular polygons under different compression values with the constraints given.

Measured Values of the Side Length Ratio Between Regular and Compressed Polygons						
Compression Level	The Type of Regular Polygon					
	Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon
Level 1 of Compression	1	1	0.4	0.5624	0.256	0.4
Level 2 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1	0.733	0.8
Level 3 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1

Fig. 5. The data table depicts the measured values of the side length ratio between the two regular polygons under different compression values with the constraints given.

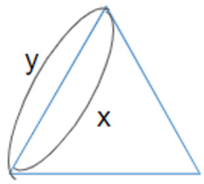

Fig. 7. This is a regular (equilateral) triangle (n = 3). Since no diagonals can be drawn from this figure, the side length ratio is x/y = 1.

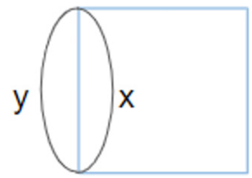

Figure 8: This is a square (n = 4). Since drawing the diagonals of this figure does not form another square, the side length ratio is x/y = 1.

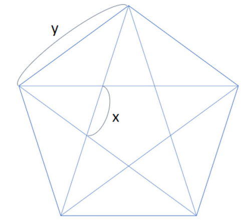

Figure 9: This is a pentagon (n = 5). Since drawing the diagonals of this figure does form one smaller pentagon inside (shown in the figure), the side length ratio between this smaller pentagon and the original pentagon is x/y = 2-Φ (approx. 0.382).

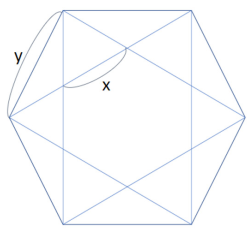

Figure 10: This is a hexagon (n = 6). Since drawing the diagonals of this figure does form one smaller hexagon inside (shown in the figure), the side length ratio between this smaller hexagon and the original hexagon is x/y = $\frac{\sqrt{3}}{3}$ (approx. 0.577).

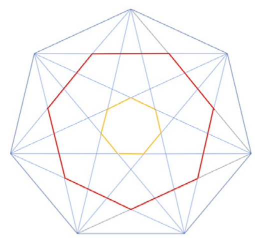

Figure 11: This is a heptagon (n = 7). Since drawing the diagonals of this figure does form two smaller heptagons inside (shown in the figure in red and yellow), the side length ratio between the smaller heptagon outlined in yellow and the original heptagon is x/y = $4\cos^2\left(\frac{\pi}{7}\right)$ -3 (approx. 0.247) and the side length ratio between the smaller heptagon outlined in red and the original heptagon is x/y = $2\cos\left(\frac{\pi}{7}\right)$ - $\sec\left(\frac{\pi}{7}\right)$ (approx 0.692).

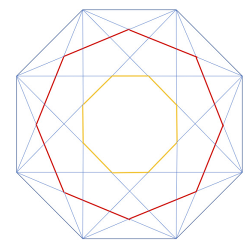

Figure 12: This is an octagon (n = 8). Since drawing the diagonals of this figure does form two smaller octagons inside (shown in the figure), the side length ratio between the smaller octagon outlined in yellow and the original octagon is x/y = $\sqrt{2} - 1$ (approx. 0.414) and the side length ratio between the smaller octagon outlined in red and the original octagon is x/y = $\sqrt{2 - \sqrt{2}}$ (approx. 0.765).

Diagrams

(see figures 7-12)

Discussion

Assessing the Trends

The data collected from this study suggests that the ratios between the side lengths of the interior regular polygon to the original regular polygon fluctuates, with even-numbered-sided polygons having a higher ratio than odd-numbered-sided polygons. For every consecutive pair of regular polygons (triangle and square, pentagon and hexagon, heptagon and octagon), a smaller polygon, similar to the polygon in which the diagonals were drawn to form that smaller polygon, is formed. Some trends show that as the compression level increases, the ratio between the polygons also increases as shown for the pentagon, hexagon, heptagon, and octagon ratios. As the number of sides increases for a given compression level, the side length ratio overall decreases. However, beyond the first 2 polygons for a given compression level, as they are the control groups for those regular polygons, the ratio fluctuates as it increases from an odd-numbered polygon to an even-numbered polygon and decreases from an even-numbered polygon to an odd-numbered polygon.

Potential Errors

Our study may have been hampered by the generalizations of certain properties of diagonals in each of the regular polygons that were analyzed, from the equilateral triangle to the regular octagon. Additionally, since this study only analyzed figures from the triangle to the octagon, there may be observations or other properties of the side length ratio between the compressed polygon and the original regular polygon beyond the octagon (nonagon, decagon, and other polygons that have more than eight sides) that may have been overlooked.

Future Research

A future research opportunity may focus on developing a relationship between the diagonals of any polygon or three-dimensional solid, not just regular, two-dimensional polygons. This could be tested by finding the ratio of the figures by determining similar properties within these figures and generalizing properly for all polygons and 3D shapes. These discoveries may prove to be useful to determine more obscure designs in architecture and origami. Knowing any polygon or 3D side length ratios would help determine how the diagonals within these shapes would compress these irregular figures into smaller and smaller figures.

Applications and Implications

A pragmatic application for this particular study would be in the field of origami. Many algorithms and folding techniques exist to construct regular polygons out of simple pieces of square paper. Using these techniques would help making constructions of regular polygons much easier and faster to execute.

Another pragmatic application for this particular study would be in the field of architecture. Using these side length ratios of compressed polygons, architects can make scaled drawings of the buildings they want to build.

Another pragmatic application for this particular study would be in the field of medicine. Angioplasties - a procedure used to clear plaque and other substances that blocks a coronary artery using a balloon - are known to be a painful medical procedure because of the size of the balloon used to flatten the plaque in the artery. By making the balloon such that it is roughly a polygonal shape and scaling it down to the ratios found in this study, the balloon would be small enough for the procedure to be effective to clear and flatten the plaque in the coronary artery, yet be less painful than using the larger balloon.

Conclusion

Polygons are everywhere in nature, and this experiment was designed to help show how diagonals can help compress a polygon down to a certain ratio to minimize the lengths by simply using its diagonals. It was hypothesized that if the number of sides of a polygon is related to the ratio between a regular polygon and a similar polygon formed from non-opposite, diagonal lines of the original regular polygon, then the ratio will change as the amount of sides change based on a certain relation and formula. This will happen because since the polygons in question are all regular, all sides and angles are defined, exploiting many properties that help develop a formula for the ratio between one side of the regular compressed polygon to one side of the original polygon. Within this experiment, the side length ratios between the compressed and the original regular polygons were collected, and a formula was generalized using fundamental mathematical principles. Based on the formula that was made for this ratio and the data that was collected, the initial hypothesis was supported. This is because since the polygons were regular, drawing diagonal lines within the polygons produce similar figures by simple angle chasing, meaning that a ratio between these figures is attainable.

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Appendix

Proof: Relationships Between Regular Polygons Based on Non-Opposite Diagonal Lines of a Regular Polygon to Form Similar, Inner Polygons

Introduction

We will discuss the relationship between regular polygons that are formed from intersecting, non-opposite diagonals. This proof will use properties of regular polygons such as properties of equiangular and equilateral angles and sides, respectively, and use circular geometry. In the end, a formula will be created to connect the ratio of the overall original regular polygon to a similar, smaller regular polygon based on the original polygon's diagonals formed from the intersection of 2 vertices.

The Proof. In order to find the ratio between 2 similar regular polygons, one that is formed from drawing diagonals between the vertices of the other polygon, we start off by looking at a part of the overall polygon.

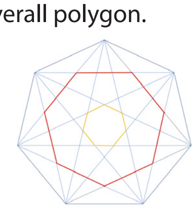


Fig. 1a. Heptagon

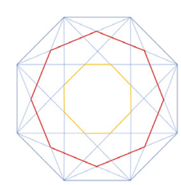


Fig. 1b. Octagon

Figures 1a and 1b: Diagrams of a heptagon (figure 1a) and an octagon (figure 1b) with their diagonals drawn to show the similar smaller polygons, shown in yellow (smallest similar polygon) and red (second smallest similar polygon).

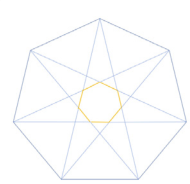


Fig. 2a. Heptagon

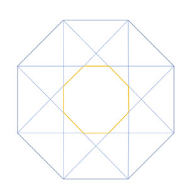


Fig. 2b. Octagon

Figures 2a and 2b: Diagrams of the smallest possible similar polygon formed from the diagonals of a heptagon (figure 2a) and an octagon (figure 2b), shown in yellow.

The regular heptagon (figures 1a and 2a) and regular octagon (figures 1b and 2b) are shown with their diagonals drawn. Each color represents a different smaller, similar regular polygon to the original regular polygon. We can isolate one of the diagonals that form one side of the similar, smaller polygon. Using this diagonal, we can use various relationships to find the desired formula.

Proof that the Similar, Smaller Polygon is Attainable and Always Exist. Let alpha (α) be equal to any of the interior angles of the n-sided polygon equal to $(180 - \frac{360}{n})°$. Drawing 2 diagonal lines, each from one vertex to another, that are positioned with points being exactly 1 vertex away from the other line, we see that an angle forms between the lines at a point of intersection. Since all regular polygons are cyclic, we can calculate the angle using circle properties. Since the lines intercept 1 arc on both sides of the line on the polygon, we see that this angle is simply half the sum of the arcs, which is equal to exactly one arc. Each arc is $\frac{1}{n}$ of the overall circle for a n-sided polygon. Therefore, the arc is exactly $\frac{360}{n}$ °. Using the definition of supplementary angles, we see that the angle supplementary to the following angle is $(180 - \frac{360}{n})°$ which is precisely α (figures 3a and 3b), the angle of each interior angle of the polygon.

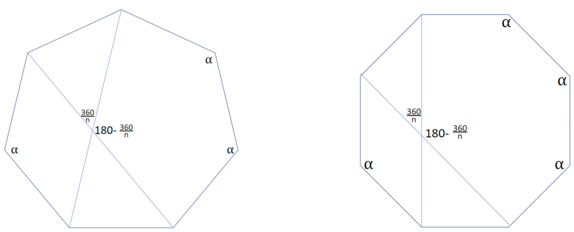


Fig. 3a. Heptagon **Fig. 3b.** Octagon

Figures 3a and 3b: Diagrams of 2 diagonals of the polygon intersecting 2 arcs on both sides of the line to form one vertex of the smaller, similar polygon.

Repeating this n times for each possible intersection of 2 arcs from 2 chords in a circle, we get a polygon with interior angles of α, which is the original overall regular polygon but is dilated by a certain scale factor. We also know these angles form another similar polygon since they will form a closed shape with only angles α. We know that it also has to be a closed shape since otherwise it would mean that it would intersect at only one possible point. By rotational symmetry, the only point within a polygon that has no other symmetry point is the center of the polygon. We know it can't lie on the center of the polygon since diagonals of an odd polygon do not intersect at the center and we restrict diagonals of even polygons to not intersect the opposite vertices of the polygon. Therefore, the shape created is not a point, but a 2D closed figure. In order for the shape to be closed, the angles must add up to 180(n - 2). Therefore, it must occur n times, resulting in an n-sided similar polygon.

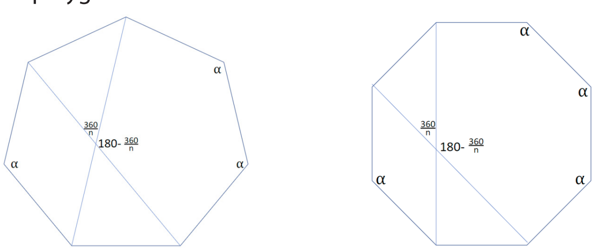


Fig. 4a. Heptagon **Fig. 4b.** Octagon

Figures 4a and 4b: Diagrams of 2 diagonals of the polygon intersecting 2 arcs on both sides of the line to form one vertex of the smaller, similar polygon.

Proof that β = θ. Using one of the diagonals from a side of the similar polygon, we let theta (θ) and beta (β) be the angles formed from the difference of an overall interior angle of a vertex and the remaining angle formed from the diagonal and the smaller portion of the polygon as shown in figures 4a and 4b. We can prove that β equals θ by the property that all regular polygons are cyclic. Since they are all cyclic, the polygon's vertices all lie on the circle. Since β and θ intercept the same arc length, by the property of inscribed angles, β = θ.

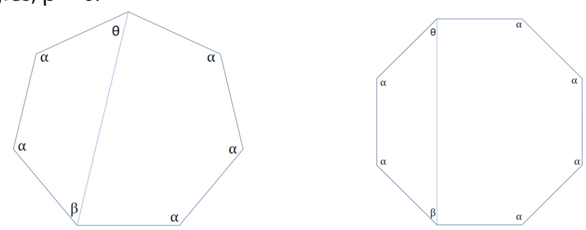


Fig. 5a. Heptagon **Fig. 5b.** Octagon

Figures 5a and 5b: Diagrams of the angles of β and θ within the heptagon (figure 5a) and octagon (figure 5b).

Finding θ for Even and Odd Polygons. To find the overall values of θ, we can break the possible θ values into 2 cases: even and odd values.

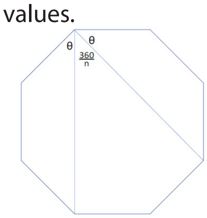


Figure 6: Diagram of one of the interior angles broken into 3 separate angles for an even-sided polygon, in this case, k = 1 is shown in an octagon since the diagonals intersect the polygon at one vertex and 2 others that are spaced 2 sides apart.

For the even-sided polygons, we can draw 2 congruent diagonal lines They will intersect at one vertex and intersect the polygon again at vertices that are 2k sides apart, where $\{k \quad Z, \frac{n}{2} - 1 \geq k \geq 1\}$ as shown in figure 6 for k = 1. By rotational symmetry, if the polygon with the original diagonal was rotated in such a way that one of the vertices would be in the place of the other vertex in the original diagram, the other diagonal would be formed. Therefore, the angle between the exterior of the inscribed angle and the side length of the newer diagonal is θ as well. We see that the inscribed angle intercepts 2k sides of the polygon. This is true since the diagonal lines must be equal by rotational symmetry, so every possible 2 sets of diagonals from one vertex must be spaced apart, starting with the opposite vertex and each shifting left or right one vertex to have congruent lines, and they can't be formed from the intersection of opposite vertices of the polygon since that would have the intersection of all the diagonals to be one point, not a closed shape. Because of this, the closest distance between the 2 diagonal lines must be 2 sides apart to be as close to intersecting the opposite vertex, and every other set of diagonals are spaced 2k sides apart. Since the angle of the triangle intercepts 2k arcs, it is exactly $\frac{360k}{n}°$, since each arc from one vertex to another is $\frac{1}{n}$ of the overall circle, the central angle is exactly $\frac{360}{n}°$ and an inscribed angle of that arc would be exactly half of the central angle of $\frac{180}{n}°$. With 2k arcs intercepted, the angle is multiplied by a factor of 2k to get $\frac{360k}{n}°$. Therefore, the following equation is true: $\alpha = 2\theta + \frac{360k}{n}$. Since $\alpha = (180 - \frac{360}{n})°$, we can substitute α in terms of n to give the equation: Solving for θ in terms of n, giving us the equation:

$180 - \frac{360}{n} = 2\theta + \frac{360k}{n}$. We can justify the inequality boundaries since k can't be less than 1 since it has been established that those diagonals form the smallest possible polygon, and k can't be greater than $\frac{n}{2} - 1$ since that would have a negative θ value, which can't happen.

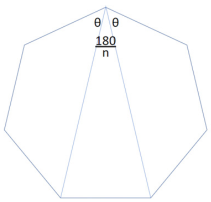


Fig. 7. Diagram of one of the interior angles broken into 3 separate angles for an odd sided polygon, in this case, k = 1 is shown in a heptagon is shown since the diagonals intersect the polygon at on vertex and 2 others that are spaced 1 side apart.

For the odd sided polygons, we can draw 2 congruent diagonal lines. They will intersect at one vertex and intersect the polygon again at vertices that are 2k - 1 sides apart, where $\{k \quad Z, \frac{n-1}{2} \geq k \geq 1\}$ as shown in figure 7 for k = 1. By rotational symmetry, if the polygon with the original diagonal was rotated in such a way that one of the vertices would be in the place of the other vertex in the original diagram, the other diagonal would be formed. Therefore, the angle between the exterior of the inscribed angle and the side length of the newer diagonal is θ as well. We see that the inscribed angle intercepts 2k - 1 sides of the polygon. This is true since the diagonal lines must be equal by rotational symmetry, so every possible 2 sets of diagonals from one vertex must be spaced apart, starting with the opposite vertex and each shifting left or right one side to have congruent lines, and they can't be formed from the intersection of opposite vertices of the polygon since no opposite vertex for any given vertex exists for odd sided polygons. Because of this, the closest distance between the 2 diagonal lines must be 1 side apart to be as close to intersecting the opposite vertex and every other set of diagonals is spaced 2k - 1 vertices apart. From this, we can solve for θ. Since the angle of the triangle intercepts 2k - 1 arcs, it is exactly $\left(\frac{360k - 180}{n}\right)^\circ$, since each arc from one vertex to another is $\frac{1}{n}$ of the overall circle, the central angle is exactly and an inscribed angle of that arc would be exactly half of the central angle of $\left(\frac{180}{n}\right)^\circ$. We can now make an equation to find θ. We can form the equation: $\alpha = 2\theta + \frac{360k - 180}{n}$. We can substitute α in terms of n to give us $180 - \frac{360}{n} = 2\theta + \frac{360k - 180}{n}$. Solving for θ gives us the equation: $\theta = \left(90 - \frac{90(2k+1)}{n}\right)^\circ$. We can justify the inequality boundaries since k can't be less than 1 because it has been established that those diagonals form the smallest possible polygon, and k can't be greater than $\frac{n-1}{2}$ since that would have a negative θ value, which can't happen.

Solving for the General Formula. To find the ratio, as said before, the diagonals of vertices that are as close as possible to forming diagonals with vertices opposite to each other, but not necessarily. With that, we can draw 2 lines in which the space between the 2 lines on the original diagonal line will form a side of the smaller polygon as shown in figures 8a and 8b.

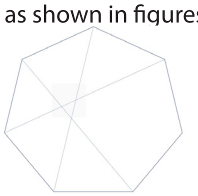


Fig. 8a. Heptagon

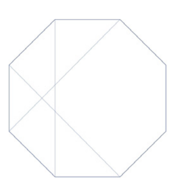


Fig. 8b. Octagon

Figures 8a and 8b: 3 congruent lines drawn inside the heptagon (figure 8a) and octagon (figure 8b) are shown.

We can now use the law of sines to solve for the smaller side, let x be the length of the smaller polygon, y be the length of the normal polygon, z be the length of the diagonal up to the point of one of the diagonals intersecting the original diagonal and w be the length of the overall diagonal as shown in figures 9a and 9b.

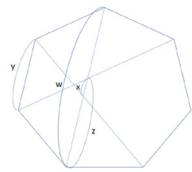


Fig. 9a. Heptagon

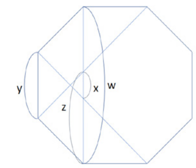


Fig. 9b. Octagon

Figures 9a and 9b: Diagrams of the heptagon (figure 9a) and octagon (figure 9b) with lengths w, x, y, and z as shown.

We can use 2 triangles to solve for the overall formula. First, we use a triangle with sides z and y, and the other with sides w and y.

The Case of the Triangle with Sides z and y. We know that one of the angles is θ, but we must know the other 2 angles to use the law of sines. By rotational symmetry, we can draw 2 diagonals that are congruent to the original one, but rotated one vertex to the right or left for both diagonals (figure 8a). Since these diagonals have the same properties as the original by rotational symmetry, they also form the sides of a triangle. Since both diagonals follow this, the angle of the triangle is exactly 180 - α since the polygon has an angle measure α. That means that using the property that all triangles' angles sum up to 180 degrees, we get that the final angle is α - θ.

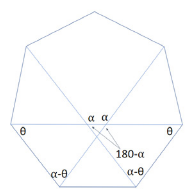


Fig. 10a. Heptagon

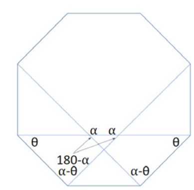


Fig. 10b. Octagon

Figures 10a and 10b: Diagrams of the heptagon (figure 10a) and octagon (figure 10b) with specified angles described shown.

Using the law of sines, we see that $\frac{sin(\alpha - \theta)}{z} = \frac{sin(180 - \alpha)}{}$. Solving for z in terms of y gives us the following equation: $z = \frac{y(sin(\alpha - \theta))}{sin(180 - \alpha)}$. Since sin(180 - m) = sin(m), we can simplify the equation to: $z = \frac{y(sin(\alpha - \theta))}{sin(\alpha)}$.

The Case of the Triangle with Sides w and y. We can use the second triangle with sides w and y to find w (figures 11a and 11b). Since one of the angles is θ and the other angle is an inscribed angle intercepting one arc, we see that the angles of the triangle are θ, $\frac{180}{n}$, and $180 - \left(\theta + \frac{180}{n}\right)$.

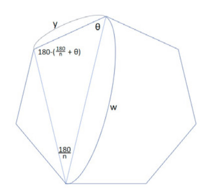


Fig. 11a. Heptagon

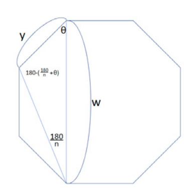


Fig. 11b. Octagon

Figures 11a and 11b: Diagrams of the triangle with lengths w and y within the heptagon (figure 11a) and octagon (figure 11b) with their angles shown.

With this, we can use the law of sines again.

$\frac{sin(\frac{180}{n})}{y} = \frac{sin(180-(\theta+\frac{180}{n}))}{w}$. Solving for w in terms of y gives us the equation: $w = \frac{y(sin(180-(\theta+\frac{180}{n})))}{sin(\frac{180}{n})}$

The Overall Formula. Now we notice that if we add z twice, it won't necessarily form w since there is an overlap between the z values of the triangles. That overlap happens to be x. Therefore, 2z-x=w. Substituting in z and w gives the equation: $2(\frac{y(sin(\alpha-\theta))}{sin(\alpha)})$ - x $= \frac{y(sin(180-(\theta+\frac{180}{n})))}{sin(\frac{180}{n})}$. Solving for x in terms of y gives the equation: $x = y[\frac{2sin(\alpha-\theta)}{sin(\alpha)} - \frac{sin(180-(\frac{180}{n}+\theta))}{sin(\frac{180}{n})}]$. Therefore, the ratio of the smaller polygon to the larger polygon is $\frac{x}{y} = \frac{2sin(\alpha-\theta)}{sin(\alpha)} - \frac{sin(180-(\frac{180}{n}+\theta))}{sin(\frac{180}{n})}$. Since $\alpha = 180 - \frac{360}{n}$, we can substitute that into the equation, giving us: $\frac{x}{y} = \frac{2sin(180-(\frac{360}{n}+\theta))}{sin(180-\frac{360}{n})} - \frac{sin(180-(\frac{180}{n}+\theta))}{sin(\frac{180}{n})}$. Since sin(180 - m)=sin(m),

$$\frac{x}{y} = \frac{2sin(\frac{360}{n}+\theta)}{sin(\frac{360}{n})} - \frac{sin(\frac{180}{n}+\theta)}{sin(\frac{180}{n})}$$

with angles measured in degrees and the θ values varying whether the polygon is even or odd. $\theta = (90 - \frac{180(1+k)}{n})°$ for even-sided polygons and $\theta = (90 - \frac{90(2k+1)}{n})°$ for odd sided polygons. Converting to radians gives the alternative form:

$$\frac{x}{y} = \frac{2sin(\frac{2\pi}{n}+\theta)}{sin(\frac{2\pi}{n})} - \frac{sin(\frac{\pi}{n}+\theta)}{sin(\frac{\pi}{n})}$$

with $\theta = \frac{\pi}{2} - \frac{\pi(1+k)}{n}$ for where $\{k \quad Z, \frac{n}{2} - 1 \geq k \geq 1\}$ for even-sided polygons in degrees or radian values and $\theta = \frac{\pi}{2} - \frac{\pi(2k+1)}{2n}$ for odd sided polygons where $\{k \quad Z, \frac{n-1}{2} \geq k \geq 1\}$ for degrees and radian values.

Table of Exact/Approximate Values Based on the Formula

Values of $\frac{x}{y}$

Exact/Approximate Values of the Side Length Ratio Between Regular and Compressed Polygons						
Compression Level	The Type of Regular Polygon					
	Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon
Level 1 of Compression	1	1	2-Φ (approx. 0.382)	$\frac{\sqrt{3}}{3}$ (approx. 0.577)	$4cos^2(\frac{\pi}{7})$ -3 (approx. 0.247)	$\sqrt{2}-1$ (approx. 0.414)
Layer 2 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1	$2cos(\frac{\pi}{7})$ - $sec(\frac{\pi}{7})$ (approx. 0.692)	$\sqrt{2-\sqrt{2}}$ (approx. 0.765)
Layer 3 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1

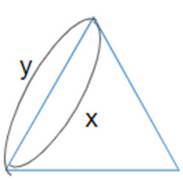
(Phi or Φ is defined to be the golden ratio or $\frac{1+\sqrt{5}}{2}$)

Table of Measured Values

Values of $\frac{x}{y}$

Measured Values of the Side Length Ratio Between Regular and Compressed Polygons						
Compression Level	The Regular Polygon					
	Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon
Level 1 of Compression	1	1	0.4	0.5624	0.256	0.4
Level 2 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1	0.733	0.8
Level 3 of Compression	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	DNE, angle is out of boundaries	1	1

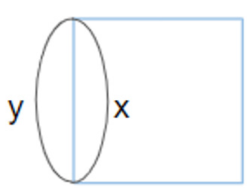
Diagrams For k=1



Triangle

Measured: $\frac{x}{y} = 1$

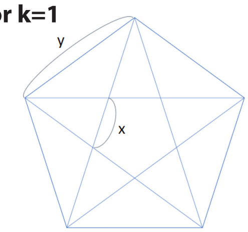
Calculated: $\frac{x}{y} = 1$



Square

Measured: $\frac{x}{y} = 1$

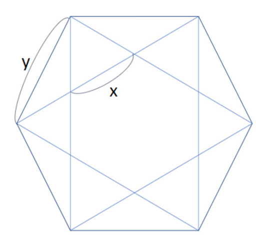
Calculated: $\frac{x}{y} = 1$



Pentagon

Calculated: $\frac{x}{y} = 2 - \Phi \approx 0.382$

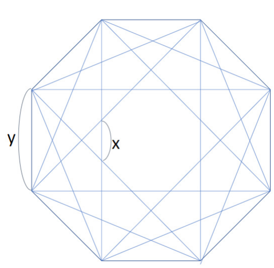
Measured: $\frac{x}{y} = 0.4$



Hexagon

Calculated: $\frac{x}{y} = \frac{\sqrt{3}}{3} \approx 0.577$

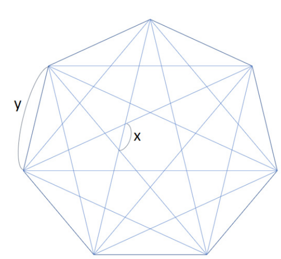
Measured: $\frac{x}{y} = 0.256$



Octagon

Calculated: $\frac{x}{y} = \sqrt{2} - 1 \approx 0.414$

Measured: $\frac{x}{y} = 0.4$



Heptagon

Calculated: $\frac{x}{y} = 4cos^2(\frac{\pi}{7}) - 3 \approx 0.247$

Measured: $\frac{x}{y} = 0.256$